On Diffeomorphism Groups of Surfaces

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Cal Poly

A Master's Thesis in Mathematics

Introduction

Question. What are the symmetries of an object X?

X could be a set, a group, a topological space,...

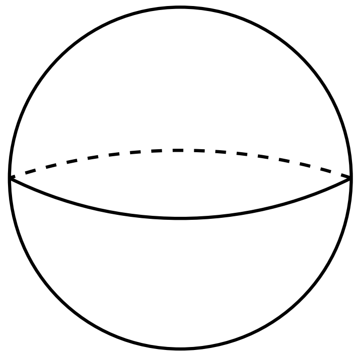
A *symmetry* of X is a bijection $X \to X$ that preserves some desired structure of X.

Example. If $X = \{1, 2, 3\}$, a symmetry of X is a permutation of these 3 elements.

The symmetries of X form a group under composition.

Example. Consider the 2-sphere S^2.

$$S^2 = \left\{x \in \mathbb{R}^3 : ||x|| = 1\right\}.$$



Geometric viewpoint: The symmetries of S^2 are *isometries*, bijections preserving Euclidean distance.

Every* isometry of S^2 is a rigid rotation of $\mathbb{R}^3$. So $\mathrm{Isom}(S^2) = \mathrm{SO}(3)$.

Fix an orthonormal basis of $\mathbb{R}^3$. Then $\mathrm{SO}(3)$ is representable as the set of 3×3 matrices with real entries of determinant 1.

Question. *But what is the shape of* $\mathrm{SO}(3)$*?*

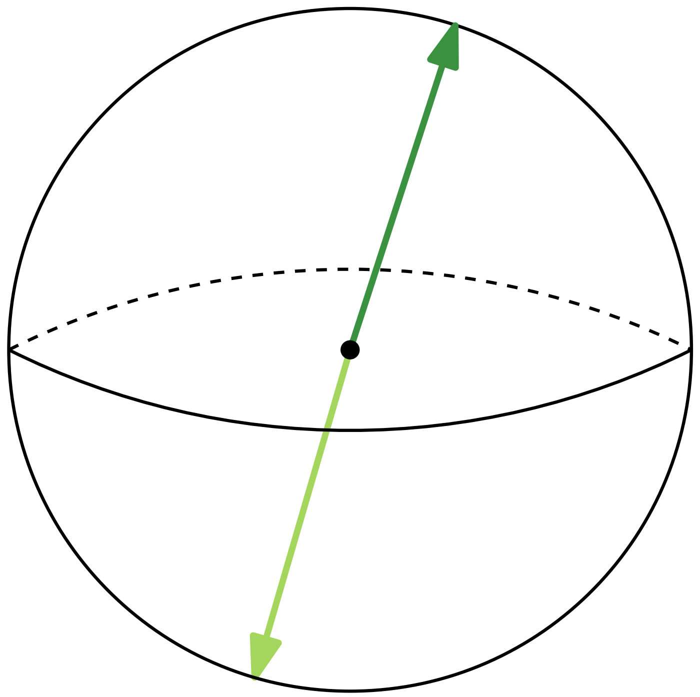
* orientation-preserving

The *shape* of $\mathrm{SO}(3)$

Every rotation is uniquely determined by an axis and an amount to rotate by, from 0 to π radians.

Consider the solid ball of radius π.

$\pi B^3 = \left\{x \in \mathbb{R}^3 : ||x|| \leq \pi\right\}.$



Rotate about the axis determined by v by $||v||$ radians using the right hand rule.

If $v \in \pi B^3$ has $||v|| = \pi$, then v and $-v$ determine the same rotation of $\mathbb{R}^3$.

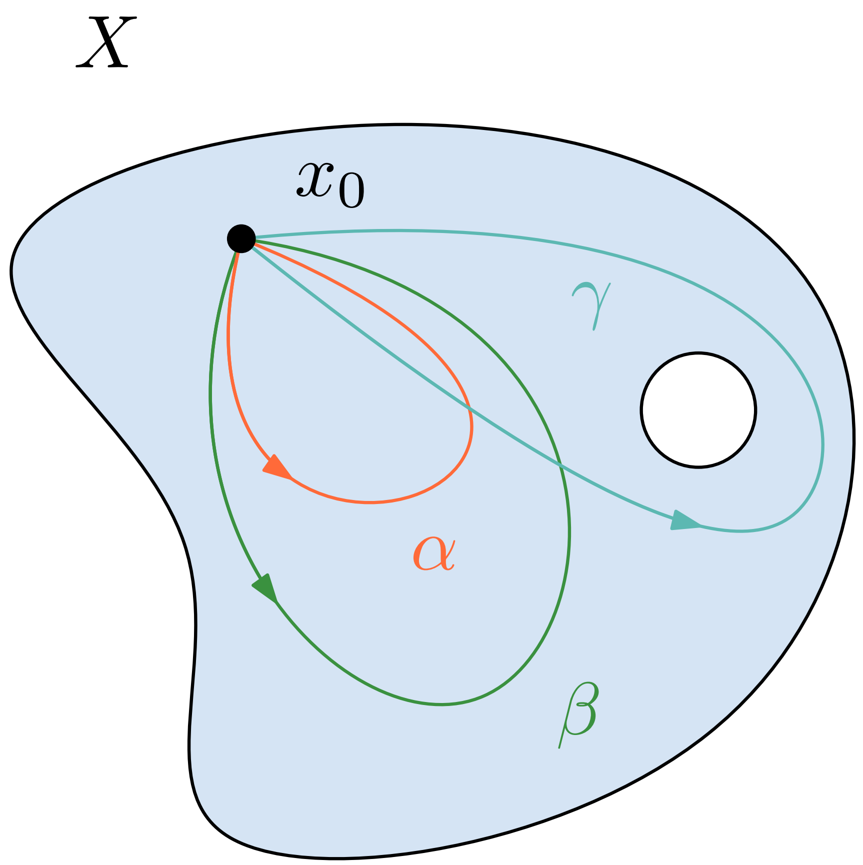
So $\mathrm{SO}(3)$ is the quotient space of πB^3 obtained by identifying antipodal points on the boundary of πB^3.

This quotient space is $\mathbb{R}P^3$, 3-dimensional real projective space.

We can understand the symmetries of S^2 by understanding the *shape* of its isometry group $\mathrm{SO}(3)$.

The *n-th homotopy groups* characterize the n-dimensional holes in a space.

Let X be a space, and fix a baspoint $x_0 \in X$.



Consider the set of loops in X based at x_0.

Two loops α and β based at x_0 are *path homotopic* if α can be continuously deformed to β, keeping x_0 fixed.

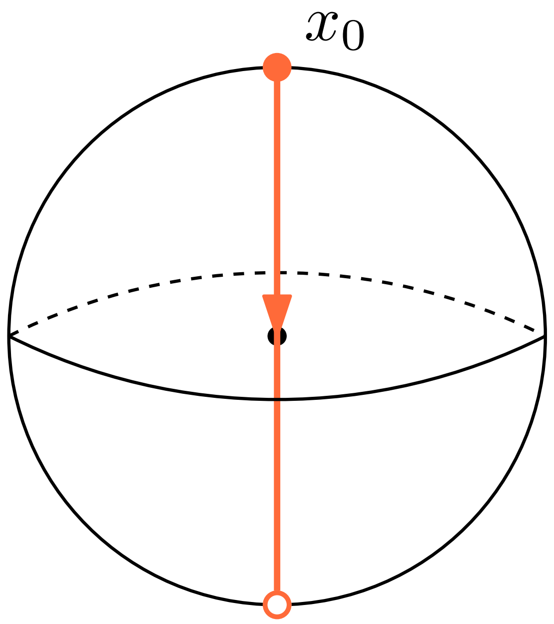
The loops α and β are homotopic to the constant loop at x_0.

But γ is not homotopic to the constant map to x_0, so γ is nontrivial. This is capturing the hole in X.

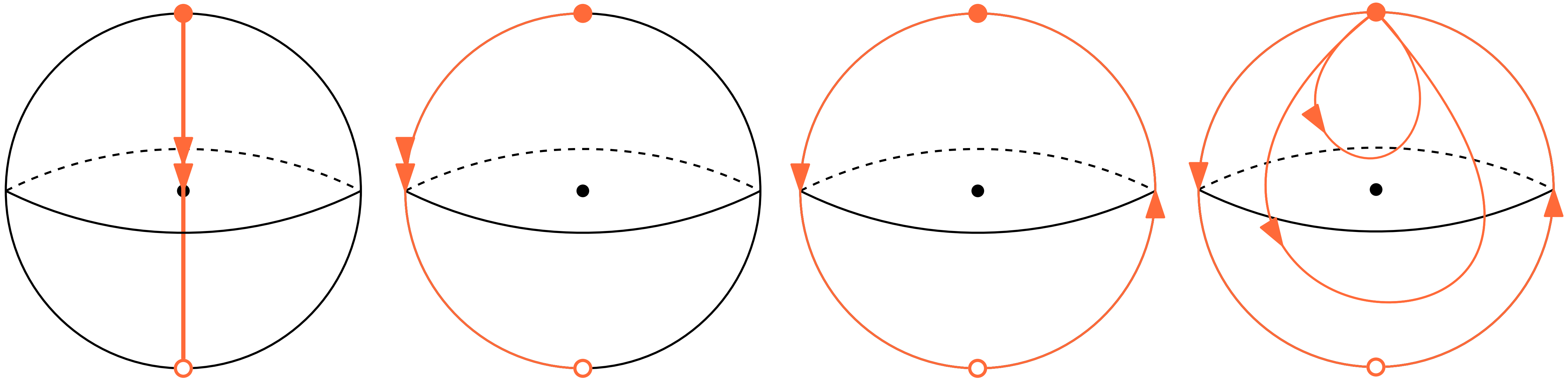
The *fundamental group* $\pi_1(X, x_0)$ is the set of homotopy classes of loops in X based at x_0. One defines π_n in a similar fashion.

The *shape* of $\mathrm{SO}(3)$

A nontrivial loop γ



γ has order 2: γ^2 is contractible



Traverse γ twice

Push γ^2 to surface

Reflect one copy of γ to antipodal side

Contract the loop on the surface

γ is the only nontrivial loop in $\mathrm{SO}(3)$ based at x_0, up to homotopy.

So $\pi_1(\mathrm{SO}(3), x_0) \cong \mathbb{Z}/2\mathbb{Z}$, the group of order 2.

$$\pi_n(\mathrm{SO}(3), x_0) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n = 2 \\ \mathbb{Z} & n = 3 \end{cases}$$

We now understand $\mathrm{SO}(3)$, the group of *orientation-preserving* isometries of S^2.

If we allow *orientation-reversing* isometries, then $\mathrm{Isom}(S^2) = \mathrm{O}(3)$.

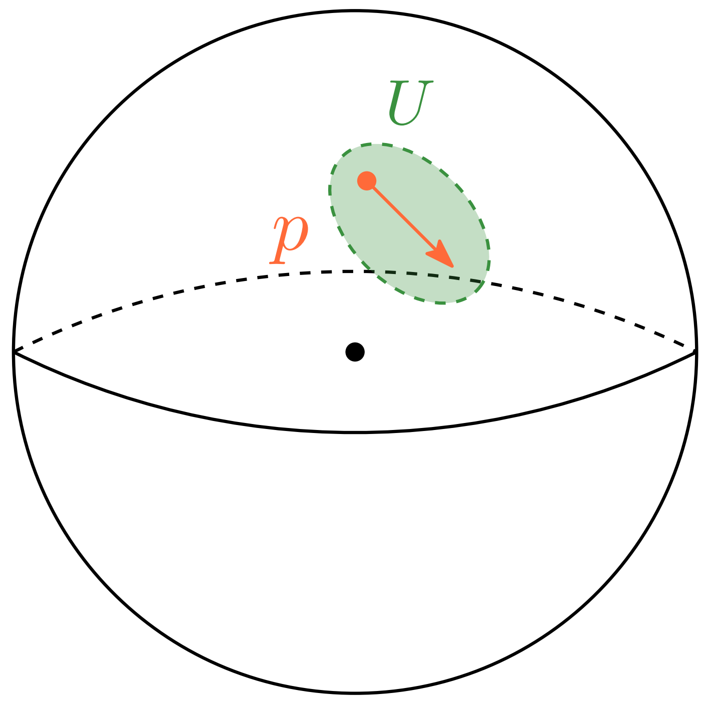
$\mathrm{O}(3)$ is representable as the set of 3×3 real matrices with determinant ± 1, so $\mathrm{O}(3)$ consists of two copies of $\mathrm{SO}(3)$.

This holds in general: $\mathrm{O}(n+1)$ is the isometry group of $S^n = \left\{x \in \mathbb{R}^{n+1} : ||x|| = 1\right\}$.

Question. *What happens if we remove the restriction that symmetries of S^2 must be isometries?*

We now require that symmetries of S^2 preserve the structure of S^2 as a *smooth manifold.*

Such maps are called *diffeomorphisms*, smooth maps with smooth inverses. They form the *diffeomorphism group* $\mathrm{Diff}(S^2)$.



Every isometry of S^2 is a diffeomorphism, so $\mathrm{O}(3) \subset \mathrm{Diff}(S^2)$.

Example: point-pushing. U a neighborhood of a point $p \in S^2$. f the identity on $S^2 \setminus U$, f pushes p within U.

There exists a one-parameter family of diffeomorphisms of S^2 between f and id, so f is in the identity path component of $\mathrm{Diff}(S^2)$.

The isometry of S^2 given by $-I_3 \in \mathrm{O}(3)$ is not isotopic to the identity. So $\mathrm{O}(3)$ is not path-connected.

Question. Did we obtain "more" symmetries of S^2 by considering diffeomorphisms, not just isometries?

Answer. In some sense, no! Why? $\mathrm{O}(3)$ is a deformation retract of $\mathrm{Diff}(S^2)$.

Conjecture (Smale)**.** Let $n > 0$. Then $\mathrm{O}(n+1)$ is a deformation retract of $\mathrm{Diff}(S^n)$.

The conjecture holds in dimensions $n < 4$.

- $n = 1$ is a standard exercise.
- $n = 2$ was proved by Smale (1959).
- $n = 3$ is a result of Hatcher (1983).

The conjecture fails in dimensions $n \geq 4$.

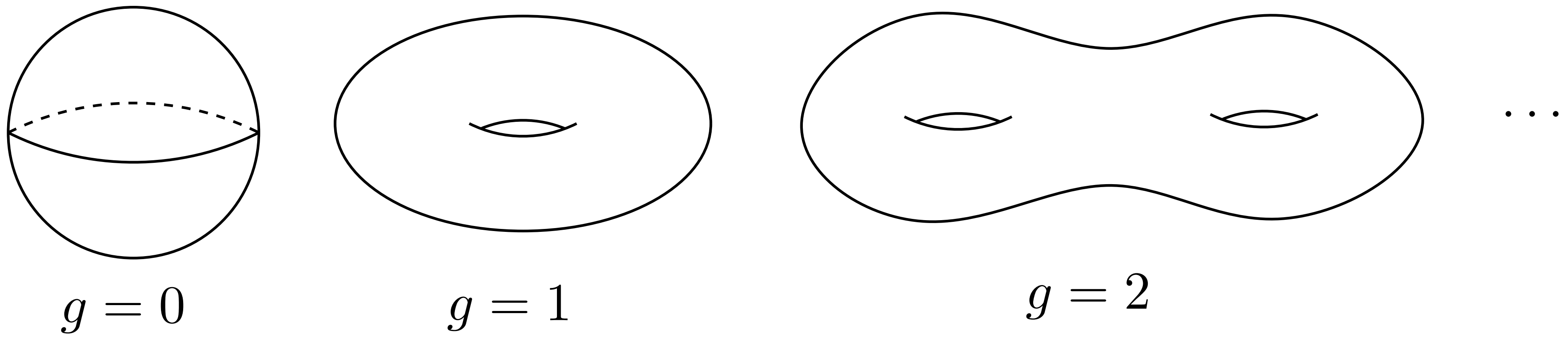
- $n > 4$ was disproved by a larger body of work, see Hatcher (2012).
- $n = 4$ was disproved by Watanabe (2018) and Gabai, Gay, and Hartman (May 2025).

Question. Let S be a closed surface. Is $\mathrm{Isom}(S)$ a deformation retract of $\mathrm{Diff}(S)$?

Connected closed surfaces are completely classified.

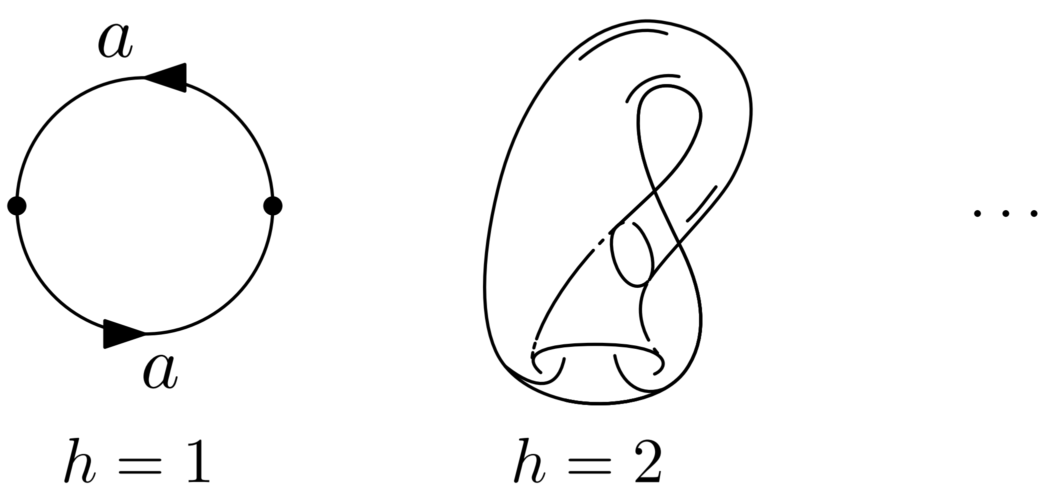
The orientable surface Σ_g of genus g:

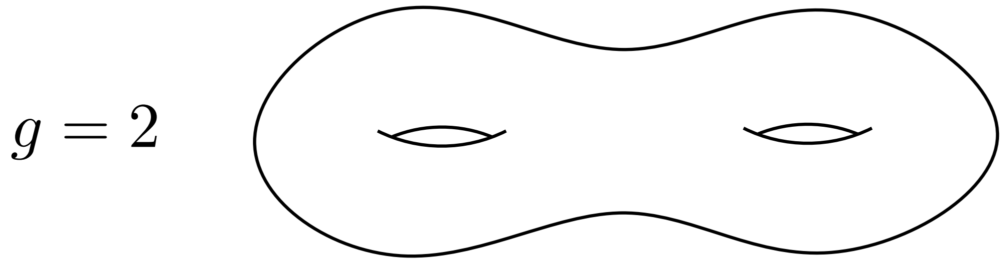
$$\Sigma_g = \#_g T$$



The nonorientable surface N_h of genus h:

$$\mathrm{N}_h = \#_h P^2$$





Let Σ_g be a orientable closed surface of genus $g \geq 1$.

Σ_g has a *Riemannian metric* which has constant curvature on Σ_g.

- $g = 1$: This metric has constant zero curvature on the torus, so the torus is flat.
- $g > 1$: This metric has constant negative curvature, so Σ_g is hyperbolic.

Let $\mathrm{Isom}(\Sigma_g)$ be the group of bijections of Σ_g which preserve this Riemannian metric.

Theorem (Hurwitz, 1892)**.** If $g > 1$, then $\mathrm{Isom}(\Sigma_g)$ is finite.

If $\mathrm{Isom}(\Sigma_g)$ is a deformation retract of $\mathrm{Diff}(\Sigma_g)$, then the components of $\mathrm{Diff}(\Sigma_g)$ are contractible.

Theorem (Earle-Eells, 1969; Gramain, 1973)**.** Let S be a compact, connected, smooth surface that is not homeomorphic to the sphere, projective plane, torus, or Klein bottle. The surface S may or may not have boundary. Then the components of $\mathrm{Diff}(S)$ are contractible.

So there are not meaningfully "more" diffeomorphisms than isometries of S.

We follow Hatcher's exposition of Gramain's 1973 proof, which uses purely topological methods.

Technical Tools

Fiber bundles

Definition. A map $p\colon E \to B$ is a *fiber bundle* with fiber $F \subset E$ if E is locally arranged as the product $B \times F$.

That is, if every point $b \in B$ has a neighborhood U such that there exists a homeomorphism $h\colon p^{-1}(U) \to U \times F$ such that this diagram commutes.

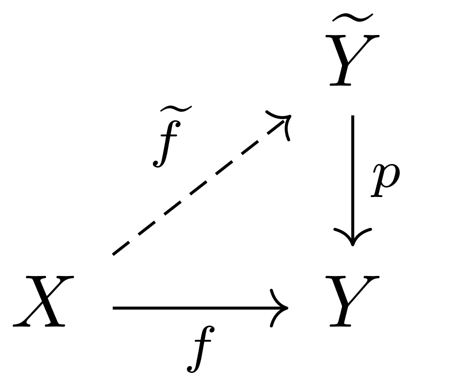
$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \pi_1} \\ & & U \end{array}$$

Write $F \to E \xrightarrow{p} B$.

We call h the *local trivialization*, B the *base space*, E the *total space*, and F the *fiber*.

Fibrations

Definition. Let $p\colon \widetilde{Y} \to Y$ be a map, and let X be a space. A *lift* of a map $f\colon X \to Y$ is a map $\widetilde{f} : X \to \widetilde{Y}$ such that $p \circ \widetilde{f} = f$.



Definition. Let $p\colon E \to B$ be a map. We say that p has the *homotopy lifting property with respect to a space* X if given a homotopy $g_t\colon X \times I \to B$ and a lift $\widetilde{g_0}\colon X \times \{0\} \to E$ of g_0, there exists a homotopy $\widetilde{g_t}\colon X \times I \to E$ lifting g_t.

The map p is a *fibration* if p has the homotopy lifting property with respect to all spaces X. A *Serre fibration* has the homotopy lifting property with respect to all n-disks.

Proposition. Every fiber bundle is a Serre fibration.

The Long Exact Sequence of a Fibration

Theorem. Suppose $p\colon E \to B$ is a Serre fibration, and choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the induced map $p_*\colon \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence, if B is path-connected, we have the long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Proved using the *long exact sequence of a pair* and *relative homotopy groups.*

Restriction Maps are Fibrations

Theorem (Palais, 1960)**.** Let M and W be smooth manifolds, and V a compact submanifold of W. Then the restriction map $\mathrm{res}_V \colon \mathrm{Emb}(W, M) \to \mathrm{Emb}(V, M)$ to V is a fiber bundle.

We study these fibrations induced by restriction using their long exact sequences.

Example. Given a smooth surface S and a point $x_0 \in S$, the evaluation map $\mathrm{ev}_{x_0}(f) = f(x_0)$ is a fibration:

$$\mathrm{Diff}(S; x_0) \to \mathrm{Diff}(S) \xrightarrow{\mathrm{ev}_{x_0}} S$$

$\mathrm{Diff}(S; x_0)$ is the set of diffeomorphisms of S which fix x_0.

$$\cdots \to \pi_n \mathrm{Diff}(S; x_0) \to \pi_n \mathrm{Diff}(S) \to \pi_n(S, x_0) \to \pi_{n-1} \mathrm{Diff}(S; x_0) \to \cdots$$

Covering Spaces

Definition. A map $p\colon \widetilde{X} \to X$ is a *covering space* if every $x \in X$ is contained in an open set U such that $p^{-1}(U)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p.

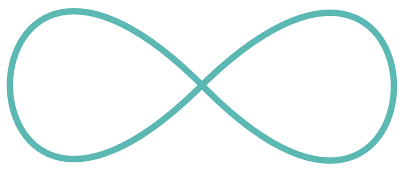
One can show that p has the homotopy lifting property with respect to all spaces.

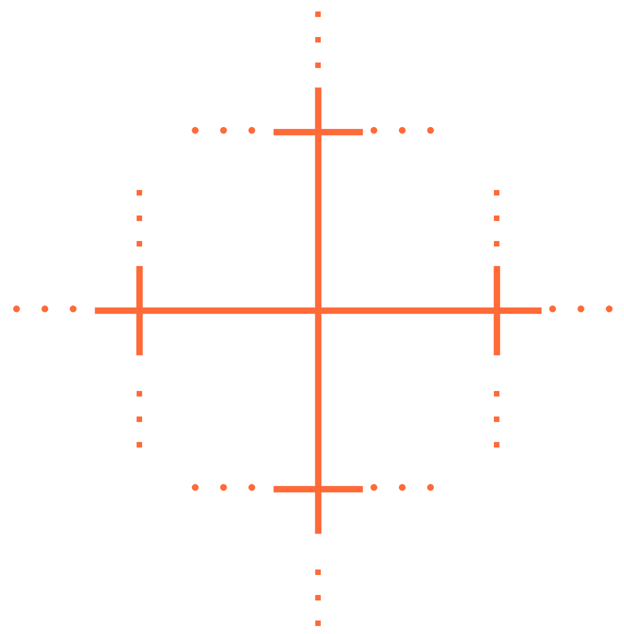
So if X is connected, p is a fiber bundle with discrete fiber.

The space $\widetilde{X}$ is *the universal cover of* X if $\widetilde{X}$ is simply connected.

Proposition. Every manifold has a universal cover.

Example. The universal cover of the figure 8 is the Cayley graph of the free group on 2 generators.





The Earle-Eells Theorem

Theorem (Earle-Eells, 1969; Gramain, 1973). Let S be a compact, connected, smooth surface that is not homeomorphic to the sphere, projective plane, torus, or Klein bottle. The surface S may or may not have boundary. Let $\mathrm{Diff}(S)$ be the group of diffeomorphisms of S which are the identity on a collar of the boundary ∂S. Then the components of $\mathrm{Diff}(S)$ are contractible.

Gramain's proof proceeds in three steps.

1) If S has no boundary, there exists a surface S_0 with boundary such that $\mathrm{Diff}(S)$ is homotopy equivalent to $\mathrm{Diff}(S_0)$.
2) The case of nonempty boundary holds, provided a certain space of arcs is contractible.
3) This arc space is contractible.

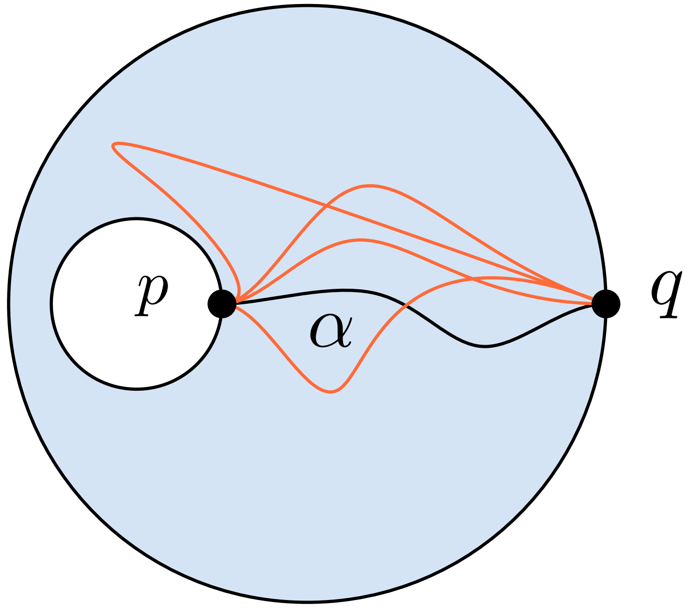
Steps 1) and 2) are proved using fibration arguments.

We will prove step 3) today.

The space of arcs

Suppose S has boundary, and pick points $p, q \in \partial S$. Let α be a proper neat arc in S from p to q. Let $\mathrm{Arc}(S, \alpha)$ be the space of all proper neat arcs in S joining p and q which are isotopic to α via an isotopy which fixes p and q.

Theorem. The space $\mathrm{Arc}(S, \alpha)$ is contractible.

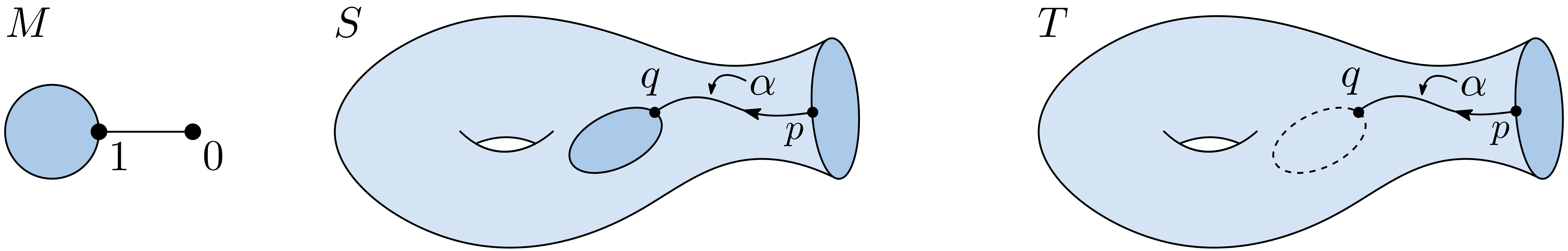


We prove the case where p and q are in different components of ∂S using fibration arguments.

We will prove the result for when p and q lie in the same boundary component using a nifty covering space argument.

Proposition. Suppose the endpoints p and q of the arc α lie in different boundary components of S. Then $\mathrm{Arc}(S, \alpha)$ is contractible.

Proof.



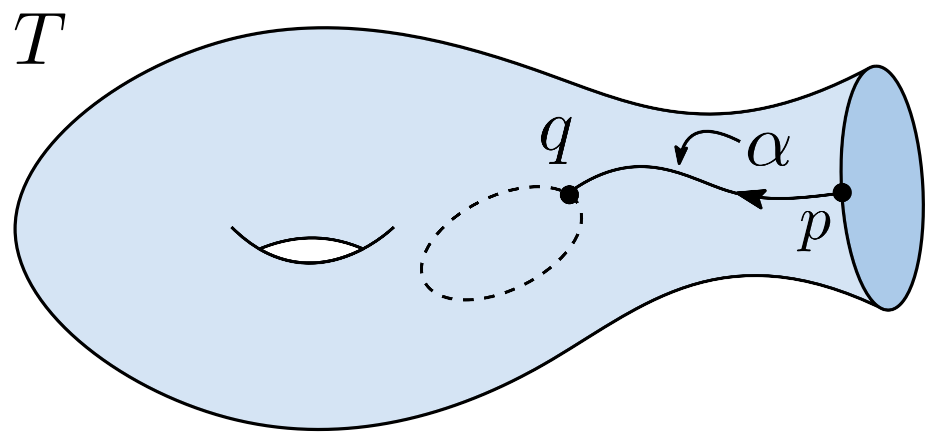
Obtain T by gluing in a disk to the component of ∂S containing q.

By Palais, we have a fibration:

$$\mathrm{Emb}((I, 0, 1, \mathrm{int}\, I), (S, p, q, \mathrm{int}\, S)) \to \mathrm{Emb}((M, 0, M \setminus 0), (T, p, \mathrm{int}\, T)) \xrightarrow{\mathrm{res}_{D^2}} \mathrm{Emb}(D^2, \mathrm{int}\, T).$$

$\mathrm{Arc}(S, \alpha)$ is a path component of the fiber, so it suffices to show that the fiber has contractible components.

We proceed by considering another fibration induced by restriction.



Due to the result of Palais, we have a fibration:

$$\mathrm{Emb}((D^2, 1, D^2 \setminus 1), (\mathrm{int}\, T, q, \mathrm{int}\, T \setminus \alpha(I))) \to \mathrm{Emb}((M, 0, M \setminus 0), (T, p, \mathrm{int}\, T)) \xrightarrow{\mathrm{res}_I} \mathrm{Emb}((I, 0, I \setminus 0), (T, p, \mathrm{int}\, T)).$$

By technical lemmas, the fiber and base space are contractible.

The long exact sequence of this fibration and Whitehead's Theorem imply that $\mathrm{Emb}((M, 0, M \setminus 0), (T, p, \mathrm{int}\, T))$ is contractible.

We have shown that $\pi_n \operatorname{Emb}((M, 0, M \setminus 0), (T, p, \operatorname{int} T)) = 0$ for all $n > 0$.

Recall the first fibration:

$$\operatorname{Emb}((I, 0, 1, \operatorname{int} I), (S, p, q, \operatorname{int} S)) \to \operatorname{Emb}((M, 0, M \setminus 0), (T, p, \operatorname{int} T)) \xrightarrow{\operatorname{res}_{D^2}} \operatorname{Emb}(D^2, \operatorname{int} T).$$

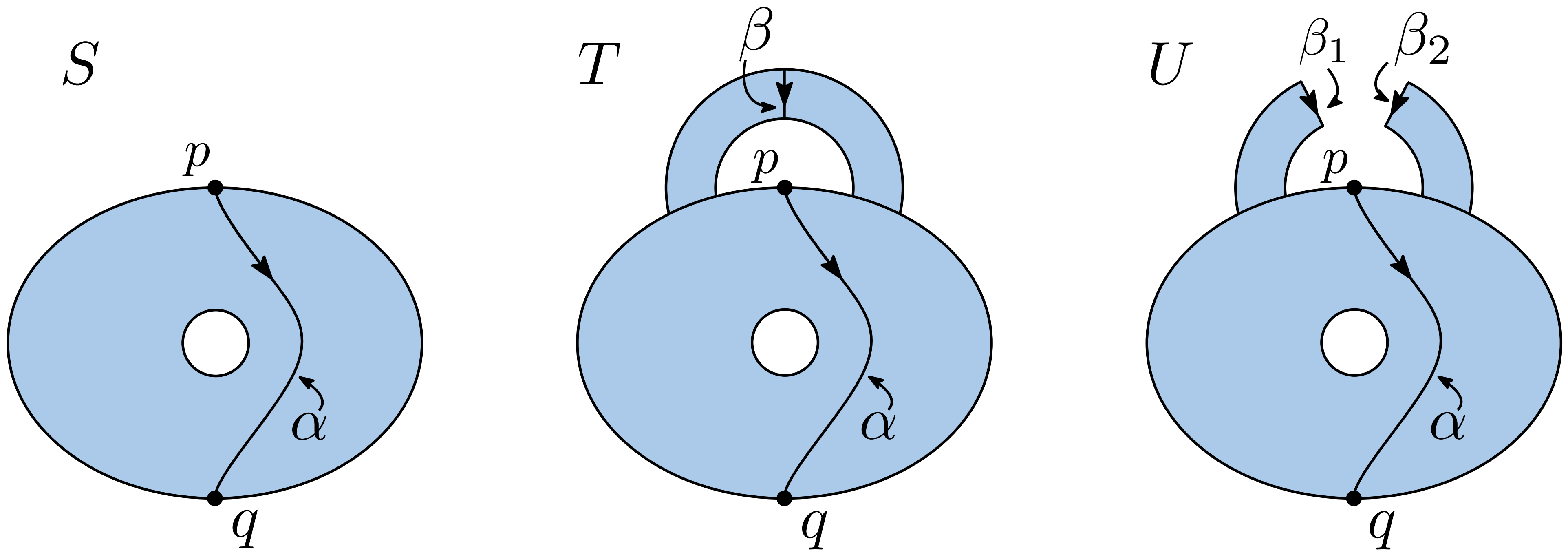
By another technical lemma, $\pi_n \operatorname{Emb}(D^2, \operatorname{int} T) = 0$ for all $n \geq 2$.

By the long exact sequence of this fibration, $\pi_n \operatorname{Emb}((I, 0, 1, \operatorname{int} I), (S, p, q, \operatorname{int} S)) = 0$ for all $n > 0$.

Whitehead's Theorem implies that $\operatorname{Emb}((I, 0, 1, \operatorname{int} I), (S, p, q, \operatorname{int} S))$ has contractible components.

Conclude that $\operatorname{Arc}(S, \alpha)$ is contractible. $\square$

Proposition. Suppose the endpoints p and q of the arc α lie in the same boundary component of S. Then $\operatorname{Arc}(S, \alpha)$ is contractible.



Proof.

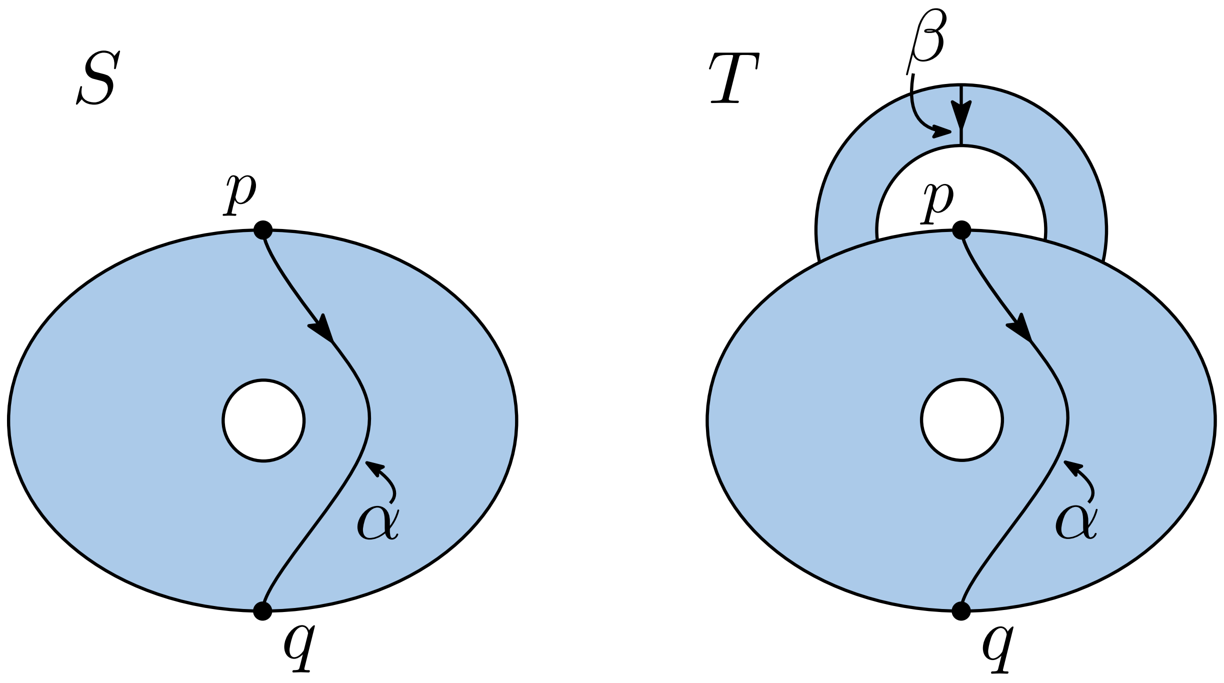
Form T by gluing a 1-handle to ∂S around p.

Cut T along β to form U, so U is homotopy equivalent to S.

Since p and q lie in different boundary components of T, $\operatorname{Arc}(T, \alpha)$ is contractible by the previous argument. So $\pi_n \operatorname{Arc}(T, \alpha) = 0$ for all $n > 0$.

We will show there exists injections $\pi_n \operatorname{Arc}(U, \alpha) \to \pi_n \operatorname{Arc}(T, \alpha)$ for all $n > 0$.

The resulting fact that $\pi_n \operatorname{Arc}(U, \alpha) = 0$ for $n > 0$ implies the space is contractible by Whitehead's Theorem.

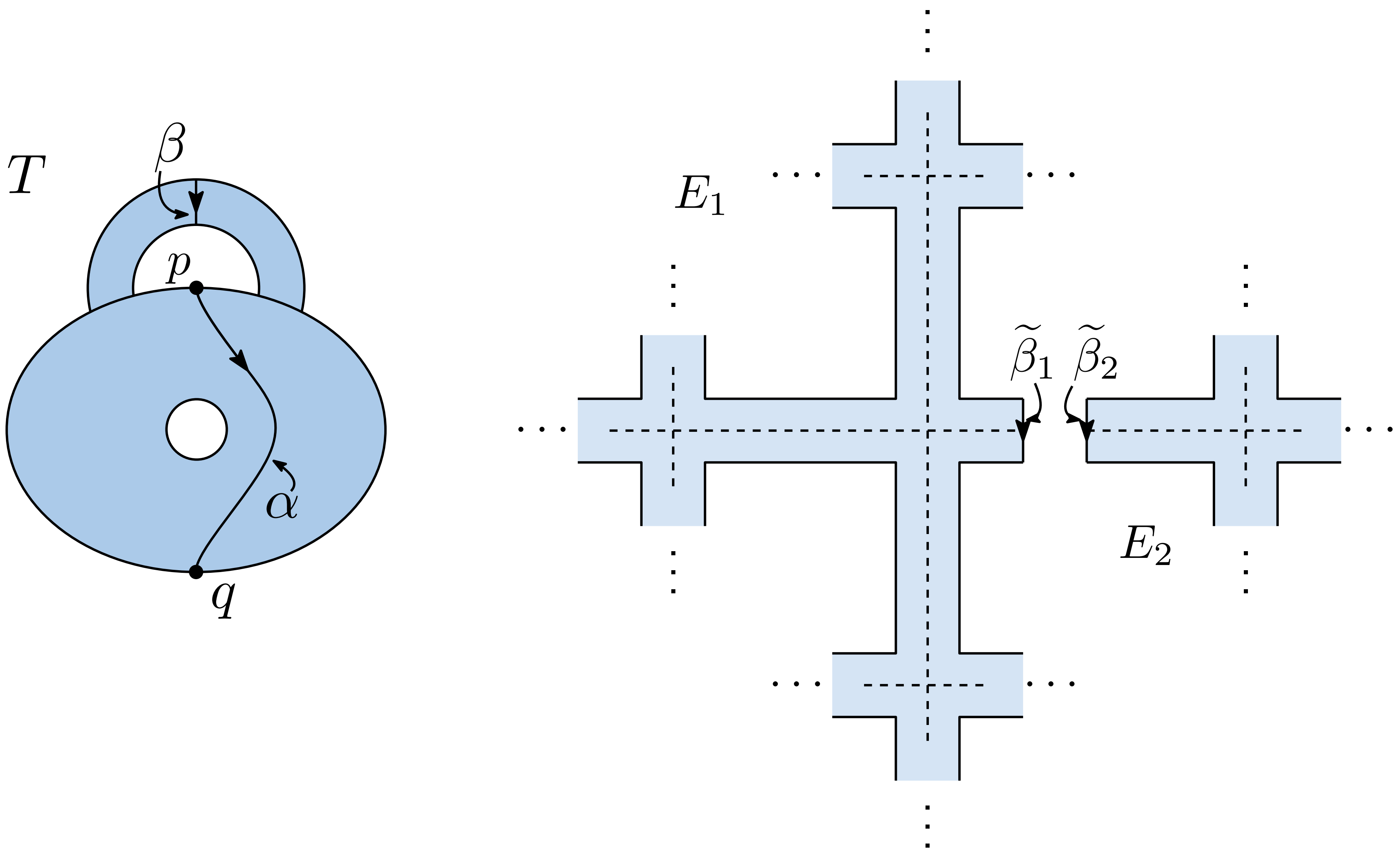


We construct this injection by considering a certain covering space of T.

Since T is homotopy equivalent to $S \vee S^1$, $\pi_1(T) \cong \pi_1(S) * \mathbb{Z}$.

We will explicitly construct the covering space $\widetilde{T}$ of T corresponding to the (conjugacy class) of the subgroup $\pi_1(S)$ of $\pi_1(T)$.

Let E be the universal cover of T.

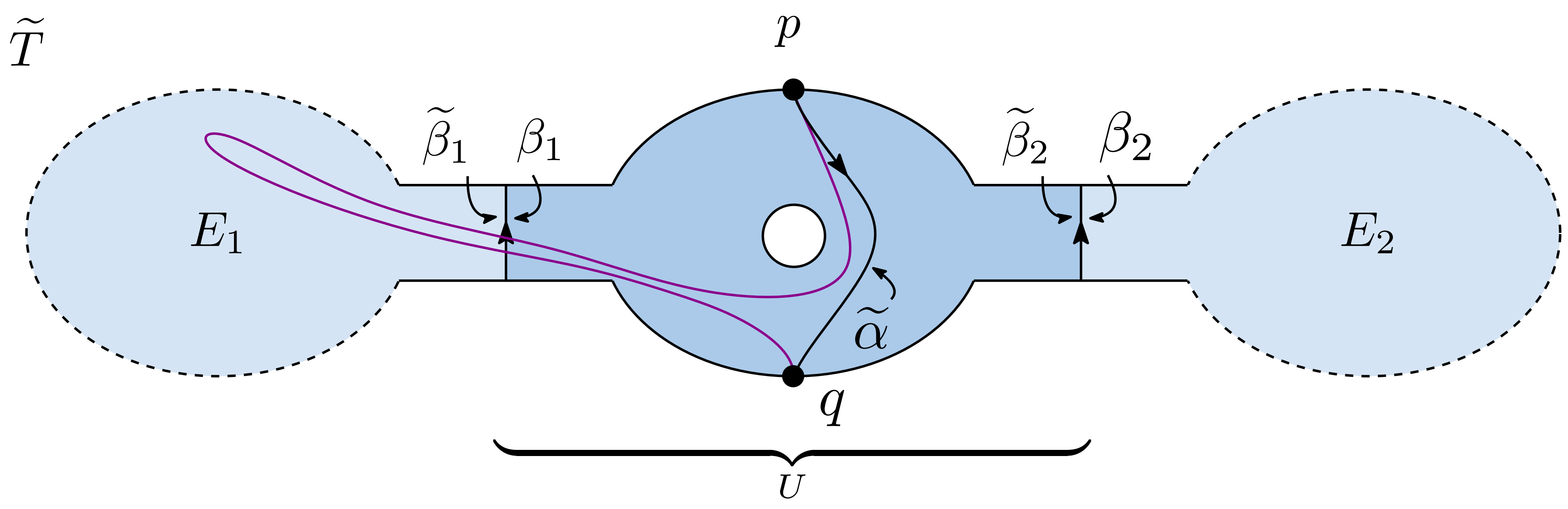


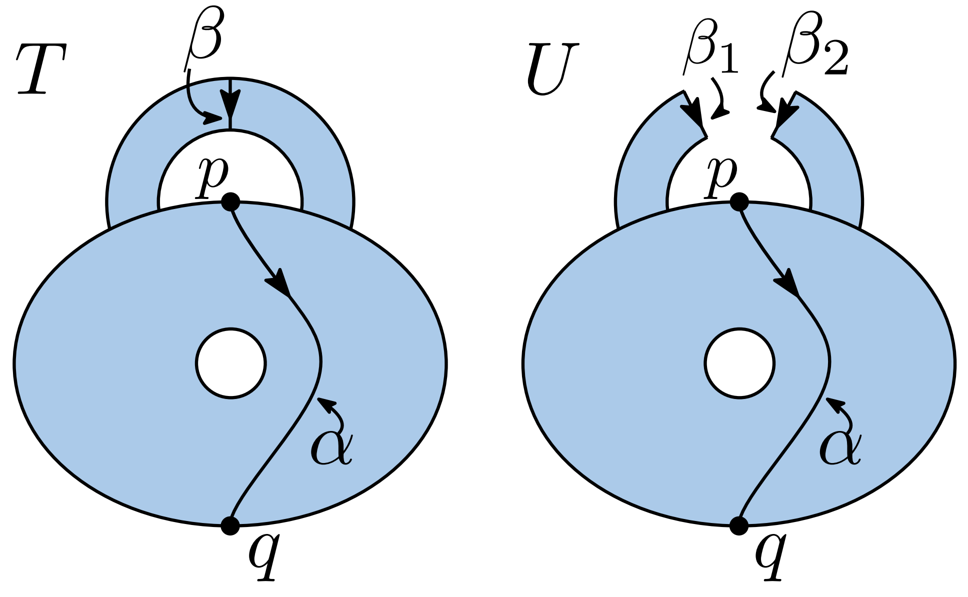
E is a "thickened" Cayley graph of the free group on n generators.

Cut along a lift $\widetilde{\beta}$ of β to obtain E_1 and E_2.

E_1 and E_2 are homeomorphic.

Form the covering space $\widetilde{T}$ of T by gluing E_1 and E_2 to U along β_1 and β_2.





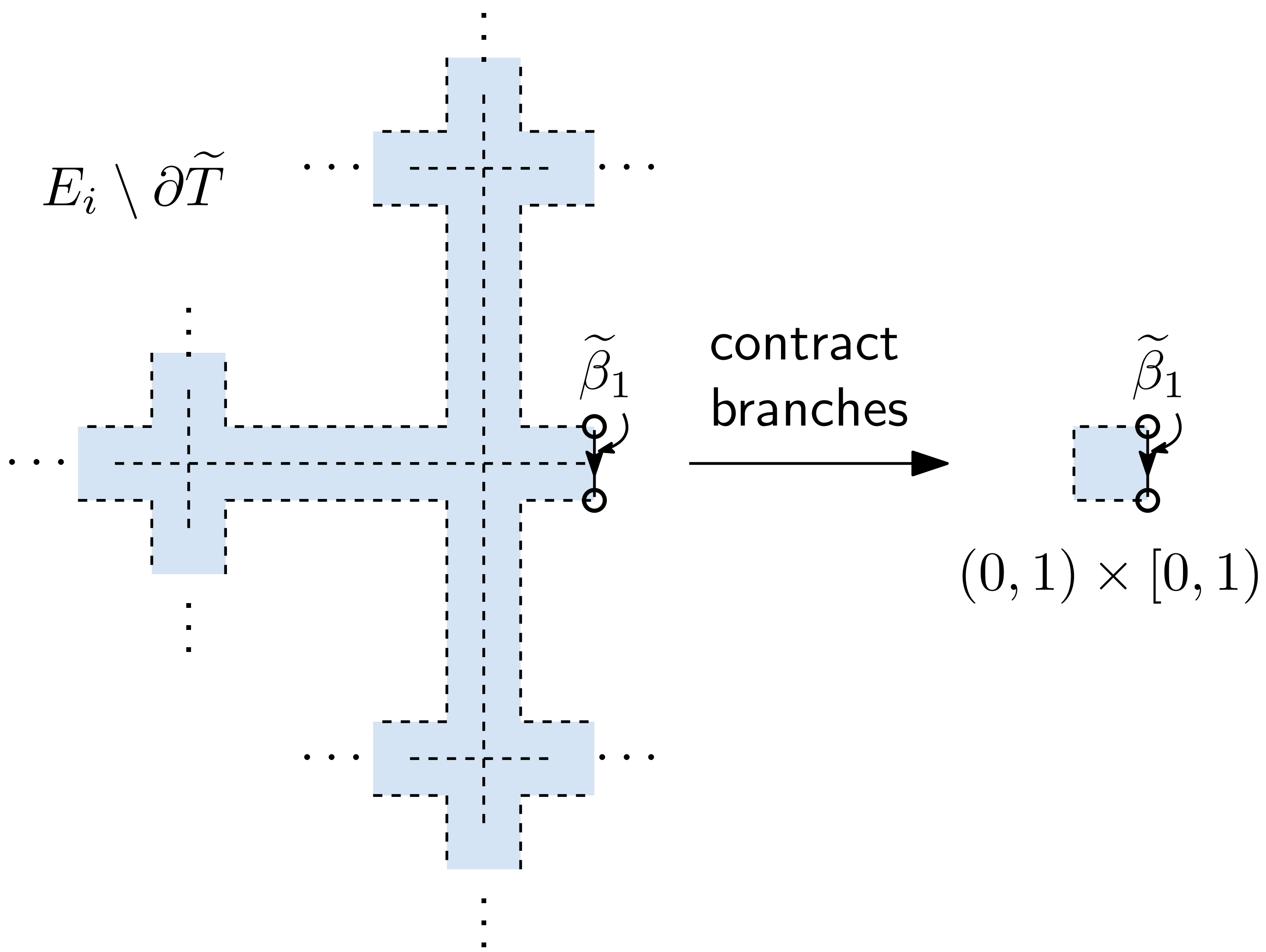
Let $\widetilde{\alpha}$ be the unique lift of α in $U \subset \widetilde{T}$.

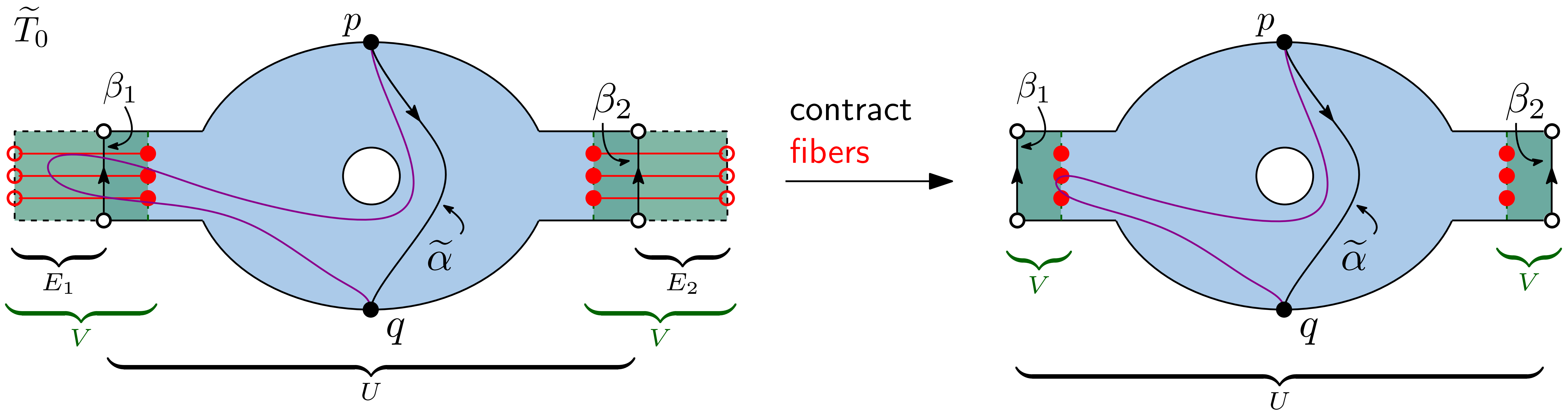
$$\operatorname{Arc}(U,\alpha) \xhookrightarrow{i_1} \operatorname{Arc}(T,\alpha) \xhookrightarrow{i_2} \operatorname{Arc}(\widetilde{T},\widetilde{\alpha}), \qquad i = i_2 \circ i_1 \colon \operatorname{Arc}(U,\alpha) \hookrightarrow \operatorname{Arc}(\widetilde{T},\widetilde{\alpha})$$

Let $i\colon \operatorname{Arc}(U,\alpha) \to \operatorname{Arc}(\widetilde{T},\widetilde{\alpha})$ be the composition. It suffices to show i induces injections on π_n.

So we define a map $r\colon \operatorname{Arc}(\widetilde{T},\widetilde{\alpha}) \to \operatorname{Arc}(U,\alpha)$ such that $r \circ i$ is homotopic to the identity.

The map r will be the restriction of the final map of an isotopy we construct.


$E_i \setminus \partial\widetilde{T}$
$\widetilde{\beta}_1$
contract
branches
$\widetilde{\beta}_1$
$(0,1) \times [0,1)$



Let $\widetilde{T}_0$ be $\widetilde{T}$ without the portion of $\partial\widetilde{T}$ lying in E_1 and E_2.

Let V be a neighborhood of $\widetilde{T}$ containing E_1, E_2, β_1, and β_2.

Isotope $\widetilde{T}_0$ into $U \subset \widetilde{T}$ by contracting the fibers $[0,1) \in (0,1) \times [0,1)$.

Define $r\colon \operatorname{Arc}(\widetilde{T},\widetilde{\alpha}) \to \operatorname{Arc}(U,\alpha)$ to be the restriction of the final map of this isotopy.

We see that $r \circ i$ is homotopic to the identity map on $\mathrm{Arc}(U, \alpha)$. Let $n > 0$.

Therefore the induced maps $(r \circ i)_*$ and id_* are equal.

$$\mathrm{Arc}(U,\alpha) \xrightarrow{i} \mathrm{Arc}(\widetilde{T},\widetilde{\alpha}) \xrightarrow{r} \mathrm{Arc}(U,\alpha), \qquad r \circ i \simeq_p \mathrm{id}$$

$$\pi_n \mathrm{Arc}(U,\alpha) \xrightarrow{i_*} \pi_n \mathrm{Arc}(\widetilde{T},\widetilde{\alpha}) \xrightarrow{r_*} \pi_n \mathrm{Arc}(U,\alpha), \qquad r_* \circ i_* = \mathrm{id}_*$$

Since id_* is an isomorphism and $(r \circ i)_* = r_* \circ i_*$, $i_* \colon \pi_n \mathrm{Arc}(U, \alpha) \to \pi_n \mathrm{Arc}(\widetilde{T}, \widetilde{\alpha})$ is injective.

$$\mathrm{Arc}(U,\alpha) \overset{i_1}{\hookrightarrow} \mathrm{Arc}(T,\alpha) \overset{i_2}{\hookrightarrow} \mathrm{Arc}(\widetilde{T},\widetilde{\alpha}), \qquad i = i_2 \circ i_1$$

$$\pi_n \mathrm{Arc}(U,\alpha) \xrightarrow{(i_1)_*} \pi_n \mathrm{Arc}(T,\alpha) \xrightarrow{(i_2)_*} \pi_n \mathrm{Arc}(\widetilde{T},\widetilde{\alpha}), \qquad i_*$$

Then i_* is injective implies $(i_1)_* \colon \pi_n \mathrm{Arc}(U, \alpha) \to \pi_n \mathrm{Arc}(T, \alpha)$ is injective.

Since $\pi_n \mathrm{Arc}(T, \alpha) = 0$, we have $\pi_n \mathrm{Arc}(U, \alpha) = 0$.

This statement holds for all $n > 0$, so by Whitehead's Theorem, $\mathrm{Arc}(U, \alpha)$ is contractible.

Since U is homotopy equivalent to S, conclude that $\mathrm{Arc}(S, \alpha)$ is contractible. □

Thank you!

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